

Can non-commutativity resolve the big-bang singularity?

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Abstract. A possible way to resolve the singularities of general relativity is proposed based on the assumption that the description of space-time using commuting coordinates is not valid above a certain fundamental scale. Beyond that scale it is assumed that the space-time has non-commutative structure leading in turn to a resolution of the singularity. As a first attempt towards realizing the above programme a modification of the Kasner metric is constructed which is commutative only at large time scales. At small time scales, near the singularity, the commutation relations among the space coordinates diverge. We interpret this result as meaning that the singularity has been completely delocalized.

1 Introduction

It is folk wisdom that singularities and divergences are not really physical but rather technical artifacts indicating the limitations of the theory in which they appear. Most theories do contain singularities as an essential element and these are generally considered to contain important information on possible extensions. Among elementary particle physicists, for example, it is widely believed that infinities in renormalizable field theories not only are not problematic since they can be treated by the renormalization procedure but instead can be considered as signals of a nearby new-physics threshold. A possible way to resolve field-theory singularities, that is, infinities related to the field-theoretical description of particle physics is to introduce still another scale, this time related to the possible unification scale of the non-gravitational interactions. Grand unified theories (GUTs) with $\mathcal{N} = 1$ supersymmetry have been constructed which can be made finite even to all loops, including the soft supersymmetry breaking sector [1–4]. There exists a method to construct GUTs with reduced independent parameters which consists of searching for renormalization-group invariant (RGI) relations valid below the Planck scale and which in turn are preserved down to the GUT scale [4, 5]. Of particular interest is the possibility of finding RGI relations among couplings which guarantee finiteness to all orders in perturbation theory. In order to reach this goal it is sufficient

to study the uniqueness of the solutions to the one-loop finiteness conditions [1, 2]. The finite unified $\mathcal{N} = 1$ supersymmetric SU(5) GUTs constructed in this way have predicted correctly the top-quark mass, for example, from the dimensionless sector (gauge–Yukawa unification) [1, 3]. The search for RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories, which involves parameters of dimensions one and two. In the SSB sector, besides the constraints imposed by finiteness there are further restrictions imposed by phenomenology. This in turn has led to a weakening of the universality of soft scalar masses at the unification point and to the introduction of a sum rule instead. In case the lightest supersymmetric particle (LSP) is a neutralino the usual Higgs mass is predicted to be in the range 115–130 GeV [4].

Here we should like to continue the above point of view and consider the singularities of general relativity as signaling a new structure of space-time and in turn as a problem whose solution can eventually be offered by non-commutative geometry. The ultimate aim is the construction of a non-commutative generalization of the theory of general relativity, which becomes essentially non-commutative in regions where the commutative limit would be singular. The physical idea we have in mind is that the description of space-time using a set of commuting coordinates is only valid at curvature scales smaller than some fundamental one. At higher scales it is impossible to localize a point and a new geometry should be used. We can think of the ordinary Minkowski coordinates as macroscopic order parameters obtained by “course-graining” over regions whose size is determined by a fundamental area scale \bar{k} , which is presumably, but

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not necessarily, of the order of the Planck area $G\hbar$. They break down and must be replaced by elements of a non-commutative algebra when one considers phenomena at higher scales.

As a first concrete example we construct a modification of the Kasner metric which is non-singular whose singularity is resolved into an essentially non-commutative structure. We recall that the singularity of the Friedmann–Robertson–Walker isotropic cosmological models, which constitute a basis for comparison of theoretical predictions with observation is not sufficiently general [6]. On the other hand the anisotropic Kasner metric connected with the description of the oscillatory approach to the cosmological singularity is considered sufficiently general [6]. In addition we recall that Heckmann–Schucking type of metrics [7] can bridge the Kasner with the Friedmann–Robertson–Walker models which are suitable for describing the later stages of the cosmological evolution.

It has been argued [9] from simple examples that a differential calculus over a non-commutative algebra uniquely determines a gravitational field in the commutative limit. Some examples have been given of metrics which resulted from a given algebra and given differential calculus [10,11]. Here we aboard the inverse problem, that of constructing the algebra and the differential calculus from the commutative metric. As an example in which the above conjectures can be tested we choose the Kasner-like metric which exhibits the most general singularity.

In the following, Greek indices take values from 0 to 3; the first half of the alphabet is used to index (moving) frames and the second half to index generators. Latin indices a, b , etc. take values from 0 to $n - 1$ and the indices i, j , etc. values from 1 to 3. More details can be found in [10].

2 The general formalism

According to the general idea outlined above a singularity in the metric is due to the use of commuting coordinates beyond their natural domain of definition into a region where they are physically inappropriate. From this point of view the space-time V should be more properly described “near the singularity” by a non-commutative algebra \mathcal{A} over the complex numbers with four hermitian generators x^λ . We suppose also that there is a set of n ($= 4$) antihermitian “momentum generators” p_α and a “Fourier transform”

$$F : x^\mu \longrightarrow p_\alpha = F_\alpha(x^\mu),$$

which takes the position generators to the momentum generators.

If the metric which we introduce is the flat metric then we shall see that $[p_\alpha, x^\mu] = \delta_\alpha^\mu$ and in this case the “Fourier transform” is the simple linear transformation

$$p_\alpha = \frac{1}{i\hbar} \theta_{\alpha\mu}^{-1} x^\mu$$

for some symplectic structure $\theta^{\alpha\mu}$; that is, an antisymmetric non-degenerate matrix. As a measure of non-

commutativity, and to recall the many parallels with quantum mechanics, we use the symbol \hbar , which will designate the square of a real number whose value could lie somewhere between the Planck length and the proton radius. If the matrix is not invertible then it is no longer evident that the algebra can be generated by either the position generators or the momentum generators alone. In such cases we define the algebra \mathcal{A} to be the one generated by both sets.

We assume that \mathcal{A} has a commutative limit which is an algebra $\mathcal{C}(V)$ of smooth functions on a space-time V endowed with a globally defined moving frame θ^α which commutes with the elements of \mathcal{A} ; that is, for all $f \in \mathcal{A}$

$$f\theta^\alpha = \theta^\alpha f. \quad (2.1)$$

We shall see that it implies that the metric components must be constants, a condition usually imposed on a moving frame.

Following strictly what one does in ordinary geometry, we shall choose the set of derivations

$$e_\alpha = \text{ad } p_\alpha, \quad e_\alpha f = [p_\alpha, f] \quad (2.2)$$

to be dual to the frame θ^α , that is, with

$$\theta^\alpha(e_\beta) = \delta_\beta^\alpha. \quad (2.3)$$

We define the differential exactly as in the commutative case. If e_α is a derivation of \mathcal{A} then for every element $f \in \mathcal{A}$ we define df by the constraint $df(e_\alpha) = e_\alpha f$.

It is evident that in the presence of curvature the 1-forms cease to anticommute. On the other hand it is possible for flat “space” to be described by “coordinates” which do not commute. The correspondence principle between the classical and non-commutative geometries can be also described as the map

$$\tilde{\theta}^\alpha \mapsto \theta^\alpha, \quad (2.4)$$

with the product satisfying the condition

$$\tilde{\theta}^\alpha \tilde{\theta}^\beta \mapsto P^{\alpha\beta}{}_{\gamma\delta} \theta^\gamma \theta^\delta.$$

The tilde on the left is to indicate that it is the classical form. The condition can be written also as

$$\tilde{C}^\alpha{}_{\beta\gamma} \mapsto C^\alpha{}_{\eta\zeta} P^{\eta\zeta}{}_{\beta\gamma} \quad (2.5)$$

or as

$$\lim_{\hbar \rightarrow 0} C^\alpha{}_{\beta\gamma} = \tilde{C}^\alpha{}_{\beta\gamma}. \quad (2.6)$$

We write $P^{\alpha\beta}{}_{\gamma\delta}$ in the form

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2} \delta_\gamma^{[\alpha} \delta_\delta^{\beta]} + i\hbar \mu^2 Q^{\alpha\beta}{}_{\gamma\delta}. \quad (2.7)$$

Flat non-commutative space is a solution to the problem of constructing a non-commutative metric, given by the choice

$$e_\alpha^\mu = \delta_\alpha^\mu, \quad K_{\alpha\beta} = -\frac{1}{i\hbar} \theta_{\alpha\beta}^{-1} \in \mathcal{Z}(\mathcal{A}). \quad (2.8)$$

We have introduced the inverse matrix $\theta_{\alpha\beta}^{-1}$ of $\theta^{\alpha\beta}$; we must suppose the Poisson structure to be non-degenerate: $\det \theta^{\alpha\beta} \neq 0$. The relations can be written in the form

$$\begin{aligned} p_\alpha &= -K_{\alpha\mu}x^\mu, \\ [p_\alpha, p_\beta] &= K_{\alpha\beta}. \end{aligned} \tag{2.9}$$

This structure is flat according to our definitions. Here is manifest one of the essential points of a Fourier transform. In the limit when the $\theta_{\alpha\beta}$ tend to zero, the points become well defined and in the opposite limit, when the $\theta_{\alpha\beta}$ tend to infinity, the momenta become well defined. In general (2.9) will be of the form

$$2P^{\alpha\beta}{}_{\gamma\delta}p_\alpha p_\beta = K_{\gamma\delta}. \tag{2.10}$$

The corresponding rotation coefficients are given by

$$C^\alpha{}_{\gamma\delta} = -4P^{\alpha\beta}{}_{\gamma\delta}p_\beta. \tag{2.11}$$

We shall find it convenient to consider a curved geometry as a perturbation of a non-commutative flat geometry. The measure of non-commutativity is the parameter \hbar ; the measure of curvature is a quantity μ^2 . We assume that $\hbar\mu^2$ is small and that in the flat-space limit we have commutation relations of the form

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}, \quad J^{\mu\nu} = \theta^{\mu\nu}(1 + O(i\hbar\mu^2)).$$

3 The commutative Kasner metric

Given a symmetric matrix $Q = (Q_b^a)$ of real numbers, one possibility for the Kasner metric is given by

$$\tilde{\theta}^0 = d\tilde{t}, \quad \tilde{\theta}^a = d\tilde{x}^a - Q_b^a \tilde{x}^b \tilde{t}^{-1} d\tilde{t}. \tag{3.1}$$

The 1-forms $\tilde{\theta}^\alpha$ are dual to the derivations

$$\tilde{e}_0 = \tilde{\partial}_0 + Q_j^i \tilde{x}^j \tilde{t}^{-1} \tilde{\partial}_i, \quad \tilde{e}_a = \tilde{\partial}_a$$

of the algebra \mathcal{A} . The Lie-algebra structure of the derivations is given by the commutation relations

$$[\tilde{e}_a, \tilde{e}_0] = \tilde{C}^b{}_{a0} \tilde{e}_b, \quad [\tilde{e}_a, \tilde{e}_b] = 0, \tag{3.2}$$

with

$$\tilde{C}^b{}_{a0} = Q_b^a \tilde{t}^{-1}.$$

We have written the frame in coordinates which are adapted to the asymptotic condition. There is a second set which is also convenient, with space coordinates x'^a given in matrix notation by

$$x'^a = (t^{-Q}x)^a. \tag{3.3}$$

The frame can then be written, again in matrix notation, with space components in the form

$$\theta^a = (t^Q d(t^{-Q}x))^a = (t^Q dx')^a.$$

The expression for $\tilde{C}^b{}_{a0}$ contains no parameters with dimension but it has the correct physical dimensions. Let G_N be Newton's constant and μ a mass such that $G_N\mu$ is a length scale of cosmological order of magnitude. As a first guess we would like to identify the length scale determined by \hbar with the Planck scale: $\hbar G_N \sim \hbar$ and so we have $\hbar \sim 10^{-87} \text{ s}^2$ and since μ^{-1} is the age of the universe we have $\mu \sim 10^{-17} \text{ s}^{-1}$. The dimensionless quantity $\hbar\mu^2$ is given by $\hbar\mu^2 \sim 10^{-120}$. In the Kasner case the role of μ is played by \tilde{t}^{-1} at a given epoch \tilde{t}_0 .

We shall see below that the spectrum of the commutator of two momenta is the sum of a constant term of order \hbar^{-1} and a "gravitational" term of order $\mu\tilde{t}^{-1} = \hbar^{-1} \times (\hbar\mu)\tilde{t}^{-1}$. So the gravitational term in the units we are using is relatively important for $\tilde{t} \lesssim \hbar\mu$. The existence of the constant term implies that the gravitational field is not to be identified with the non-commutativity per se but rather with its variation in space and time.

The components of the curvature form are given by

$$\tilde{\Omega}^a{}_0 = (Q^2 - Q)_b^a \tilde{t}^{-2} \tilde{\theta}^0 \tilde{\theta}^b, \tag{3.4}$$

$$\tilde{\Omega}^a{}_b = -\frac{1}{2} Q_{[c}^a Q_{d]b} \tilde{t}^{-2} \tilde{\theta}^c \tilde{\theta}^d. \tag{3.5}$$

The curvature form is invariant under a uniform scaling of all coordinates. The Riemann tensor has components

$$\tilde{R}^a{}_{0c0} = (Q^2 - Q)_b^a \tilde{t}^{-2}, \quad \tilde{R}^a{}_{bcd} = Q_{[c}^a Q_{d]b} \tilde{t}^{-2}.$$

The vacuum field equations reduce to the equations

$$\text{Tr}(Q) = 1, \quad \text{Tr}(Q^2) = 1.$$

If q_a are the eigenvalues of the matrix Q_b^a there is a 1-parameter family of solutions given by

$$q_a = \frac{1}{1 + \omega + \omega^2} (1 + \omega, \omega(1 + \omega), -\omega). \tag{3.6}$$

The most interesting value is $\omega = 1$ in which case

$$q_a = \frac{1}{3} (2, 2, -1).$$

The curvature invariants are proportional to \tilde{t}^{-2} ; they are singular at $\tilde{t} = 0$ and vanish as $\tilde{t} \rightarrow \infty$.

The values $q_a = c$ for the three parameters are also of interest. The Einstein tensor is given by

$$\tilde{G}_0^0 = -3c^2 \tilde{t}^{-2}, \quad \tilde{G}_b^a = -c(3c - 2) \delta_b^a \tilde{t}^{-2}.$$

For the value $c = 2/3$ the space is a flat FRW with a dust source given by

$$\tilde{T}_{00} = -\frac{1}{8\pi G_N} \tilde{G}_{00} = \frac{1}{6\pi G_N}.$$

For $c = 1/3$ the space is Einstein with a time-dependent cosmological "constant".

4 The algebra of the non-commutative Kasner metric

We are now in a position to write the algebra of the non-commutative Kasner metric. From the structure of the frame we obtain commutation relations between the position and momentum generators. Using these and Jacobi identities we determine the momentum–momentum or the position–position commutation relations.

4.1 The position–momentum relations

From the correspondence with the commutative limit of frame it is easy to see that the position–momentum commutation relations are

$$\begin{aligned} [p_0, t] &= 1, [p_0, x^b] = Q_c^b \tau x^c, \\ [p_a, t] &= 0, [p_a, x^b] = \delta_a^b. \end{aligned} \tag{4.1}$$

Note that we have introduced the element τ of the sub-algebra of \mathcal{A} generated by t which must tend to a constant multiple of t^{-1} in the commutative limit. In fact the derivations defined by (2.2) satisfy (2.3).

4.2 The momentum–momentum relations

We write the commutation relations (2.10) satisfied by the momentum generators p_α using (2.7) and the Ansatz

$$Q^{cd}{}_{a0} = \frac{1}{8} k^{(c} Q_a^{d)}, \quad Q^{cd}{}_{ab} = \frac{1}{4} k^c k^d K_{ab}.$$

We then obtain the relations

$$[p_a, p_b] = K_{ab} + L_{ab}(\tau), \tag{4.2}$$

$$[p_0, p_a] = K_{0a} + L_{0a}(\tau). \tag{4.3}$$

To be consistent with the commutative limit the L_{0a} must be given by

$$L_{0a}(\tau) = Q_a^b \tau p_b. \tag{4.4}$$

The element τ tends to a multiple of t^{-1} of the commutative algebra. We choose k^a to be an eigenvector of Q_a^b with eigenvalue q ; that is, $Q_a^b k^a = q k^b$. This choice simplifies the calculations to be performed. To further simplify we choose $k^a = (0, 0, 1)$, $K_{0a} = -\frac{1}{i\tilde{k}} l_a$, $l_a = (0, 0, l)$. Then (4.3) can be written as

$$[p_0, p_3] = K_{03} - q\tau p_3 = -\frac{1}{i\tilde{k}} l, \tag{4.5}$$

$$[p_0, p_1] = K_{01} = 0,$$

$$[p_0, p_2] = K_{02} = 0.$$

The space derivations of the element τ must vanish; that is, $[p_a, \tau] = 0$. To find an explicit form of τ we multiply (4.5) by $-i\tilde{k}\mu^2 q$ (where μ^2 is a scale of curvature as explained in Sect. 2); then we see that by introducing $\tau = -i\tilde{k}\mu^2 q p_3$ and setting $m^2 = -i\tilde{k}\mu^2 q K_{03}$ we determine

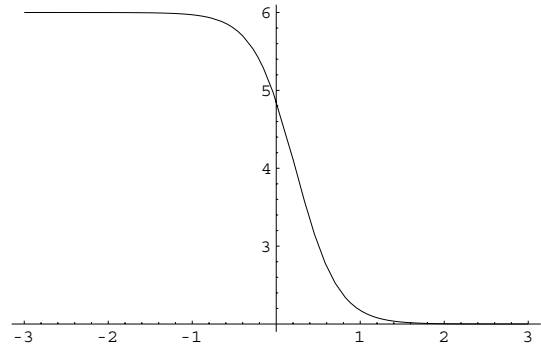


Fig. 1. A non-singular solution for τ . (Horizontal axis t in units of $(m\sqrt{|q|})^{-1}$, τ in units of $m\sqrt{|q|}$)

an element of the algebra that has the property $[p_a, \tau] = 0$ and obeys the differential equation

$$\dot{\tau} - q\tau^2 + m^2 = 0, \quad m^2 = \mu^2 q l. \tag{4.6}$$

Let us first note an interesting duality of (4.6), namely

$$\tau \rightarrow 1/\tau, \quad m^2 \rightarrow q. \tag{4.7}$$

The generic solution of this equation has the form

$$\tau = -\frac{1}{q} \frac{c_1 |m| \sqrt{q} e^{+|m| \sqrt{q} t} + c_2 (-|m| \sqrt{q}) e^{-|m| \sqrt{q} t}}{c_1 e^{+|m| \sqrt{q} t} + c_2 e^{-|m| \sqrt{q} t}}, \tag{4.8}$$

which e.g. when $c_1 = c_2$ becomes

$$\tau = -\frac{1}{\sqrt{q}} |m| \tanh(\sqrt{q} |m| t). \tag{4.9}$$

A general class of solutions of (4.8) are non-singular. A representative example is given in Fig. 1.

Depending on the value of q we may obtain another general class of periodic but singular solutions (like $\tau \sim \cot(t)$) and a representative example is drawn in Fig. 2. The function τ enters the curvature invariants [10] and in case it is smooth, the singularity problem is avoided.

From the Jacobi identities we find that L_{ab} must be of the form

$$L_{ab} = \frac{q}{m^2} K_{ab} \tau^2 \tag{4.10}$$

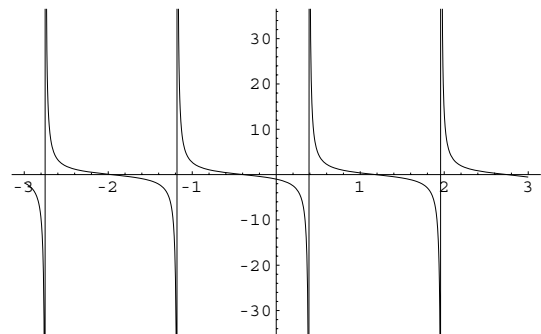


Fig. 2. A singular solution for τ . (Horizontal axis t in units of $(m\sqrt{|q|})^{-1}$, τ in units of $m\sqrt{|q|}$)

as well as the algebraic condition

$$2qK_{ab} = Q_{[a}^c K_{b]c},$$

which reduces to

$$\text{Tr}Q = -3q, \quad q_1 + q_2 + q = -3q.$$

In the following we think of Q as the matrix $Q = \text{diag}(q_1, q_2, q)$. We have determined the lowest order correction to L_{ab} , while the series does not seem to sum to a simple known function.

The momentum algebra of (2.7) takes in the present case the form

$$[p_0, p_a] = K_{0a} + Q_a^b \tau p_b, \quad (4.11)$$

$$[p_a, p_b] = K_{ab} \left(1 + \frac{q\tau^2}{\mu^2} \right). \quad (4.12)$$

The above form of the momentum algebra can be derived from the following choice for Q_{cd}^{ab} and Q_{0c}^{ab} :

$$Q_{cd}^{ab} = -i\hbar \frac{m^2}{2ql} k^a k^b K_{cd}$$

and

$$Q_{0c}^{ab} = \frac{m^2}{2l} Q_c^a k^c.$$

The commutation relations are then

$$\begin{aligned} [p_a, p_b] &= (i\hbar)^{-1} \epsilon_{abc} k^c + \frac{1}{2} p_c C^c_{ab} \\ &= (i\hbar)^{-1} \epsilon_{abc} k^c - 2i\hbar p_c p_d Q^{cd}_{ab} \\ &= (i\hbar)^{-1} \epsilon_{abc} k^c \left(1 + \frac{q\tau^2}{m^2} \right), \end{aligned} \quad (4.13)$$

$$\begin{aligned} [p_0, p_a] &= (i\hbar)^{-1} l_a + \frac{1}{2} p_c C^c_{0a} \\ &= (i\hbar)^{-1} l_a - 2i\hbar p_b p_c Q^{bc}_{0a} \\ &= (i\hbar)^{-1} l_a + \tau Q_a^d p_d. \end{aligned} \quad (4.14)$$

To simplify the commutation relations we have chosen $k^a = (0, 0, 1)$ and $l_a = (0, 0, l)$; then we have

$$\begin{aligned} [p_2, p_3] &= 0, \quad [p_3, p_1] = 0, \\ [p_1, p_2] &= (i\hbar)^{-1} \left(1 + \frac{q\tau^2}{m^2} \right), \quad [p_0, p_1] = q_1 p_1 \tau, \\ [p_0, p_2] &= q_2 p_2 \tau, \quad [p_0, p_3] = (i\hbar)^{-1} l + qp_3 \tau. \end{aligned} \quad (4.15)$$

It is possible to represent \mathcal{A} as a tensor product of two Heisenberg algebras, i.e.

$$\mathcal{A}_{12} \otimes \mathcal{A}_{30} \subset \mathcal{A}.$$

We denote the generators of \mathcal{A}_{12} by μ_1, μ_2 and the generators of \mathcal{A}_{30} by μ_0, μ_3 . Being in the system $(0, 0, 1)$ the relation between τ and p_3 becomes

$$p_3 = -\frac{1}{i\hbar\mu^2 q} \tau, \quad (4.16)$$

e.g. using (4.9) we obtain

$$p_3 = \frac{|m|}{i\hbar\mu^2 q^{3/2}} \tanh(|m|\sqrt{qt}). \quad (4.17)$$

Then we define $\mu_0 = p_0$ and $\mu_3 = \frac{1}{i\hbar} t$ and we define μ_1 and μ_2 in such a way that

$$\begin{aligned} [\mu_0, \mu_1] &= 0, \quad [\mu_3, \mu_1] = 0, \\ [\mu_0, \mu_2] &= 0, \quad [\mu_3, \mu_2] = 0. \end{aligned} \quad (4.18)$$

But

$$[\mu_0, \mu_3] \neq 0, \quad [\mu_1, \mu_2] \neq 0. \quad (4.19)$$

To achieve this we set $p_1 = \mu_1 U_1, p_2 = \mu_2 U_2$ where $U_{1,2} = U_{1,2}(t)$ and calculate the commutation relations $[\mu_0, p_{1,2}]$. We find that the U 's must satisfy the equation

$$\dot{U} = Q\tau U \quad (4.20)$$

in order for our commutation relations for μ_α to have the desired form. We denote a solution of (4.20) depending on the parameter Q as $U_{(Q)}$, i.e. $U_1 \equiv U_{(q_1)}, U_2 \equiv U_{(q_2)}$. Next calculating $[p_1, p_2]$ we find that

$$[\mu_1, \mu_2] = \frac{1}{i\hbar U_1 U_2} \left(1 + \frac{q\tau^2}{m^2} \right). \quad (4.21)$$

Similarly calculating $[\mu_0, p_3]$ we find

$$[\mu_0, \mu_3] = \frac{1}{i\hbar}. \quad (4.22)$$

4.3 The position–position relations

In the simplified momentum basis we make an ansatz for the Fourier transform relating the coordinates x, y, z, t to the momenta $\mu_1, \mu_2, \mu_0, \mu_3$:

$$\begin{aligned} x &= i\hbar\mu_1 f_1, \\ y &= i\hbar\mu_2 f_2, \\ z &= i\hbar\mu_0 f_3, \\ t &= i\hbar\mu_3, \end{aligned} \quad (4.23)$$

where $f_{1,2,3}$ are functions of t to be determined from the momentum–position commutation relations. Putting (4.23) into (4.1) we find that

$$\begin{aligned} f_1 &= U_{(q_1)}, \\ f_2 &= U_{(q_2)}, \\ f_3 &= U_{(q)}, \end{aligned}$$

where the functions U were defined above and depend on the parameter indicated in the subscript. We can now calculate the commutation relations between the coordinates and find

$$\begin{aligned} [x, y] &= i\hbar\rho, \quad [x, z] = -i\hbar q_1 \tau x U_{(q)} \\ [y, z] &= -i\hbar q_2 \tau y U_{(q)}, \quad [t, z] = i\hbar U_{(q)}, \end{aligned} \quad (4.24)$$

where

$$\rho = \left(1 + \frac{q\tau^2}{m^2}\right).$$

Then the $J^{\mu\nu}$ tensor is given by

$$J^{\mu\nu} = U_{(q)}^{-1} \begin{pmatrix} 0 & \rho U_{(q)} & -q_1 \tau x & 0 \\ -\rho U_{(q)} & 0 & -q_2 \tau y & 0 \\ q_1 \tau x & q_2 \tau y & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This can be written in the form

$$J^{\mu\nu} = S^{\mu\nu} + x^{[\mu} P^{\nu]},$$

with

$$S^{\mu\nu} = \begin{pmatrix} 0 & \rho & 0 & 0 \\ -\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P^\mu = U_{(q)}^{-1} \tau (0, 0, 0, -q_1).$$

We see then from the behaviour of τ that the commutation relations diverge at the origin. Indeed the orbital term $x^{[\mu} P^{\nu]}$ vanishes exponentially and the spin term diverges as t^{-2} .

5 Conclusions

In conclusion we propose to resolve the singularities of general relativity by assuming that space-time becomes fuzzy beyond a certain scale. In the specific example we have given here the Kasner manifold has been replaced by a non-commutative algebra, whose Jacobi identities force a modification of the time dependence of the metric. All curvature invariants depend smoothly on a element τ , which replaces the time coordinate. We have seen that particular choices of parameters in (4.8) lead to non-singular solution such as (4.9) which is a desirable result for the programme we have put forward.

We note that the above non-singular solution has the interesting property of extrapolating between two flat solutions of different (constant) commutation relations; that is, in the notation of Sect. 2,

$$\theta_-^{\alpha\beta} = \lim_{t \rightarrow -\infty} J^{\alpha\beta}$$

and

$$\theta_+^{\alpha\beta} = \lim_{t \rightarrow +\infty} J^{\alpha\beta}$$

are not equal and can be arbitrary. In particular one of them can vanish. In this way we have a smooth extrapolation between a non-commutative flat space and a commutative one. On the other hand the general solution given by (4.8) (depending on the value of q) contains periodic solutions which are singular. The duality (4.7) of (4.6) connects the singular points with the regular points.

It should be stressed though that no use is made of field equations. The restrictions on the solutions find their origin in the requirement that the non-commutative algebra be an associative one and appear as Jacobi identities on the commutation relations.

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